A prelude to Neutron Stars: The phase diagram of the strong interactions at finite density

Vikram Soni[*]

National Physical Laboratory, K.S. Krishnan Marg, New Delhi 110012, India

Dipankar Bhattacharya[†]

Raman Research Institute, Bangalore 560080, India

We consider strong interactions at finite density in mean field theory, through an effective lagrangian that can describe both nuclear matter and quark matter. This lagrangian has three couplings that are all fixed by experiment and no other parameters. With increasing baryon density we then find the following hierarchy. At nuclear density and above we have nuclear matter with chiral spontaneous symmetry breaking (SSB), followed by the pion condensed quark matter, again with chiral SSB, albeit with a different realization and finally a transition to the diquark CFL state which also has chiral SSB (and colour SSB), with yet another realization. To one's surprise at zero temperature (in mean field theory), at any finite density chiral symmetry is never restored!

We find another remarkable feature and this is that the tree level mass of the sigma particle, that is set by experiment to about 800 MeV, has a crucial and unexpected influence on the physics. Strange quark matter and strange stars are ruled out for a sigma mass above 700 MeV and neutron stars with magnetic pion condensed cores, that could provide magnetic fields of neutron stars, exist only for a small interval, between 750–850 MeV, for the sigma mass.

I. INTRODUCTION

Neutron stars have been a subject of abiding interest for several decades, almost since Landau suggested their existence, shortly after the discovery of the neutron. There are a variety of astrophysical phenomena that arise from the physics of neutron stars. The supernova explosion through which the star is born, is a spectacular luminous event. Many neutron stars work as pulsars, which generate beamed radiation in their intense magnetic fields. Neutron stars in binary systems may accrete matter from their companions, giving rise to some of the brightest X-ray sources in the sky. Some neutron stars with super-strong magnetic fields produce occassional strong bursts of gamma rays (Soft Gamma Repeaters). Most of these phenomena require us to understand the physics of matter at very high density, which govern the mass and the size of neutron stars. In other words, one needs to have a clear understanding of the equation of state of the ground state of superdense matter. Although much effort has gone into this enterprise over the last four decades it still remains poorly understood. Why?

Central densities of neutron stars are high, $\sim$5 to 10 times nuclear density $\rho_{\text{nuc}} = 0.17$ fm^{-3}. For a single species, neutrons, this naively translates into a fermi gas with typical fermi momentum, $k_f^{\text{N}} \sim 700$ MeV. On the other hand nucleons have structure and a typical size of the order of a fermi $(200 \text{ MeV})^{-1}$. It is clear that at such high densities nucleons (neutrons) cannot be treated as elementary. They are composite and resolved. They are colour singlet bound states of three valence quarks. At such high densities, therefore, treating nucleons as point particles interacting via two body (or more) forces will be inadequate. Yet most available equations of state adopt this approach and therefore fail to capture the correct physics.

On the other hand, if we use quarks as the elementary degrees of freedom, we are presently bound by the fact that only perturbative calculations can be done for QCD. This implies that calculations can be done in QCD only at very high density when the theory is approximately in an Asymptotically Free (AF) phase. However, at intermediate and low density (close to nuclear density), where a nucleonic description is valid, we cannot use perturbative QCD as the coupling becomes strong and the physics non perturbative. This is the dilemma.

There are attempts to model the physics by a two phase structure – a quark matter core with a hadronic/nucleonic exterior shell and crust. Since there is no simple way to link the two phases without using separate parameters for both, this description is somewhat arbitrary. Further, the nature of the quark matter state is not clear – for example, is it in a spontaneous chiral symmetry broken state.

Can we find a theory that can describe both these domains? We present, here, an Effective Chiral Intermediate Lagrangian, L, that has quarks, gluons and a chiral multiplet of $[\vec{\pi}, \sigma]$ that flavor-couples only to the quarks [1, 2, 3, 4, 5, 6].

This yields the nucleon as quark soliton – a bound state of quarks in a solitonic background of scalar/pseudoscalar field expectation values (EV-s) that follow from the spontaneous breaking of chiral symmetry [1, 2]. In this way we can generate nucleon matter from these nucleons at low density with a transition to quark matter at high density but with the same effective Lagrangian covering

[*]Electronic address: vsoni@del3.vsnl.net.in
[†]Electronic address: dipankar@rri.res.in

both domains. Such a unified description has not been given before and depends just on the coupling constants of the theory and not on any parameters.

To begin with let us consider the two main features of the strong interactions at low energy. These are i) that quarks are confined as hadrons and ii) chiral symmetry is spontaneously broken (SSB) with the pion as an approximate Goldstone boson. There is no specific reason that these two phenomena should occur at an identical temperature scale, though QCD lattice simulations show that for $SU_2(L) \times SU_2(R)$ they are close. The problem in giving an unequivocal answer to this question is that we are yet to find a solution to the non-perturbative aspects of QCD.

Let us consider QCD with a two flavour $SU_2(L) \times SU_2(R)$ chiral symmetry. First let us address the question of what occurs at a lower energy scale, confinement or chiral symmetry breaking. The problem is that though there is a bonafide order parameter for chiral symmetry breaking – the mass of the constituent quark, the Wilson loop is no longer an order parameter for confinement, in the presence of dynamical quarks. However, by looking at the energy density or specific heat we can get a fair idea of the change in the number of operational degrees of freedom or particle modes. Such lattice calculations indicate that the change from the large number of degrees of freedom in the quark matter phase to few degrees of

freedom in the hadronic one takes place in one broad step in temperature, indicating that the two transitions may be close for $SU_2(L) \times SU_2(R)$.

Also, if the chiral symmetry restoration (energy/temperature) scale was lower than the confinement scale we would expect hadrons to show parity doubling below the confinement scale but above the chiral SSB scale. This is not seen in finite temperature lattice simulations.

Actually, QCD can have multiple scales [5]. Apart from a confinement scale and a chiral symmetry restoration scale we also have a compositeness scale for the pion. The above considerations suggest these scales are respectively in ascending order in energy (temperature).

Let us consider the interacting fermi liquid of nucleons – nuclear matter. As the baryon density is raised beyond overlap, we expect a transition to quark matter. An interesting question arises: Is the quark matter in a chiral SSB state with constituent quarks or is it, as is usually assumed, in a chirally restored state with current quarks? As we will see the Lagrangian L, given below, answers this question [6, 7].

On the other hand there is some evidence for this intermediate Chiral Lagrangian that has simultaneously no confinement but chiral SSB. Such an effective Lagrangian has quarks, gluons and a chiral multiplet of $[\vec{\pi}, \sigma]$ that flavor couples only to the quarks.

$$L = -\frac{1}{4}G^a_{\mu\nu}G^a_{\mu\nu} - \sum \overline{\psi}\left(\not{D} + g_y(\sigma + i\gamma_5\vec{\tau}.\vec{\pi})\right)\psi - \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{1}{2}(\partial_\mu\vec{\pi})^2 - \frac{1}{2}\mu^2(\sigma^2 + \vec{\pi}^2) - \frac{\lambda^2}{4}(\sigma^2 + \vec{\pi}^2)^2 + \text{const} \quad (1)$$

The masses of the scalar (psuedoscalar) and fermions follow from the minimization of the potentials above. This minimization yields

$$\mu^2 = -\lambda^2 < \sigma >^2 \quad (2)$$

It follows that

$$m_\sigma^2 = 2\lambda^2 < \sigma >^2 \quad (3)$$

Experimentally, $< \sigma >= f_\pi$, the pion decay constant. This theory is an extension of QCD by additionally coupling the quarks to a chiral multiplet, ($\vec{\pi}$ and σ) [1, 2, 3, 4].

This Lagrangian has produced some interesting physics at the mean field level [4, 8]

(i) It provides a quark soliton model for the nucleon in which the nucleon is realized as a soliton with quarks being bound in a skyrmion configuration for the chiral field expectation values (EV) [1, 4, 8].

(ii) Such a model gives a natural explanation for the 'Proton spin puzzle'. This is because the quarks in the background fields are in a spin-isospin singlet state in which the quark spin operator averages to zero. On the

collective quantization of this soliton to give states of good spin and isospin the quark spin operator acquires a small non zero contribution [9].

(iii) Such a Lagrangian also seems to naturally produce the Gottfried sum rule [10].

(iv) Such a nucleon can also yield from first principles (but with some drastic QCD evolution), structure functions for the nucleon which is close to the experimental ones [11].

(v) In a finite temperature field theory such an effective Lagrangian also yields screening masses that match with those of a finite temperature QCD simulation with dynamical quarks [12]. This work also does not show any parity doubling for the hadronic states.

(vi) This Lagrangian also gives a consistent equation of state for strongly interacting matter at all density [4, 7, 13].

This L has a single dimensional parameter, f_π, that is the pion decay constant, and three couplings, g_3, the QCD coupling, g_y, the Yukawa coupling between quarks and mesons, that will be determined from the nucleon mass and the meson-meson coupling, λ, which, for this model, can be determined from meson meson scattering

[14]. No further phenomenological input will be used. As it stands, there is no confinement in this model, but may be dynamically generated as in QCD.

Since this is posited as an effective Lagrangian, we should have an appoximate idea of its range of validity. We find, somewhat in analogy with the top quark (large Yukawa coupling) composite higgs picture, that we can get a compositeness scale for the scalars in this model by using Renormalisation Group (RNG) evolution. We find that the wavefunction renormalisation for the scalars is inversely proportional to the running Yukawa coupling and thus naively vanishes when the Yukawa coupling blows up. For our theory such a ballpark scale falls between 700–800 MeV [15].

An independent and quite general approach in setting a limit to the range of validity of non asmptotically free (e.g. Yukawa) theories is the vacuum instability to small length scale fluctuations (or large momenta in quantum loop corrections) that plagues these theories, discovered and analytically proved by one of us [16]. The scale at which this occurs is of the same order as above. This is not very surprising since it is connected to non-AF character of the Yukawa coupling [16, 17]. This underscores the impossibility of doing loop calculations, unless we introduce a cut off, even though our L is renormalizable!

Given these facts we shall use this theory at the Mean Field level to look at different phases of this field theory. To do this we must first establish the ground state of the Baryon Number B=1 sector of this theory, i.e. the nucleon.

The plan of the paper is as follows, In Sec. 2 we review the description of the nucleon in this model, which fixes g_y. This is followed in Sec. 3 by a pedagogical description

of how we may look upon nuclear matter at density above nucleon overlap. We then look at some phases of 2 flavour quark matter in Sec. 4 and point out that the phase with lowest ground state energy is the pion condensed phase with chiral SSB. In Sec. 5 these results are generalized to 3 flavours. In Sec. 6 we review and compare the pion condensed phase with the diquark condensed colour supercondcting phase. We end this section with a review of the phase diagram of QCD at all density. In Sec. 7 we remark on the phases to be used in constructing neutron stars in a following paper and discuss the validity of Mean Field Theory (MFT).

II. THE NUCLEON IN THE CHIRAL LINEAR SIGMA MODEL WITH QUARKS

The basic fields are the three component (isospin) pion fields, $\vec{\pi}(r)$, the scalar field, $\sigma(r)$, which together form a real 4 component chiral multiplet that transforms as a representation of O(4), the rotation group in four dimensions. The fermionic fields are the quarks which transform as a fundamental representation $SU_L(L) \times SU_R(R)$ which is isomorphic to O(4). In this we ignore the gluon fields which are vector and therefore not expected to carry expectation values in MFT. However, corrections due to one-gluon exchange interactions between quarks can be included, but we shall not do so below, as their effect is small. We also neglect the pion mass corrections. The Hamiltonian [1] which is invariant under the above group transformations then reads

$$H = \int d^3x \left[\frac{1}{2}(\partial_i \vec{\pi})^2 + \frac{1}{2}(\partial_i \sigma)^2 + V(\sigma, \vec{\pi}) + \sum \psi^\dagger \left(-i\partial_i \alpha_i + \beta g_y(\sigma + i\gamma_5 \vec{\tau}.\vec{\pi}) \right) \psi \right] + O(m_\pi) \tag{4}$$

The term $V(\sigma, \vec{\pi}) = \frac{\lambda^2}{4}(\sigma^2 + \vec{\pi}^2 - f_\pi^2)^2$ is the potential functional. For the vacuum sector (no fermions) this is the quantity to be minimized. The minimum of $V(\sigma, \vec{\pi})$ occurs at $\sigma^2 + \vec{\pi}^2 = f_\pi^2$. This is the equation of a 3 sphere and thus allows for a continuous degeneracy. However, normally the choice, $<\sigma> = f_\pi, <\vec{\pi}> = 0$ is made, which is consistent with the pseudoscalar, $<\vec{\pi}>$, having a zero Vacuum Expectation Value (VEV) to avoid spontaneously violating parity in hadronic scattering. Also, with this choice the goldstone pseudoscalar excitations about the ground state are the right ones – the psuedoscalar pions. Once this choice for the vacuum state is made, it is clear that the vacuum state spontaneously violates the chiral symmetry, it changes under a O(4) rotation, even though H does not.

We now move to the description of the ground state for the the baryon number B = 1 sector.

1) The usual pattern of symmetry breaking is the space uniform VEV corresponding to the choice above.

$$<\sigma> = f_\pi, <\vec{\pi}> = 0 \tag{5}$$

This choice gives a spontaneous or Yukawa mass to the quark, $m_q = g <\sigma>$.

The lowest energy for B = 1 sector, which corresponds to the quantum numbers of the nucleon, is $M = 3m_q = 3gf_\pi$ – simply the mass of three quarks.

2) Let us now consider the case when a fermion bound state can arise from a time independent expectation value (EV) that is locally space dependent. Clearly, to have finite energy for this state requires that asymptotically (far away from the localized fermion source), the EV revert

back to the VEV above. We now move on to the so called skyrme hedgehog configuration,

$$< \sigma >= f_\pi \cos \theta(r), \quad < \vec{\pi} >= \hat{r} f_\pi \sin \theta(r) \qquad (6)$$

where $\theta(r \to \infty) = 0$, from the finite energy condition and $\theta(r \to 0) = -\pi$, for the pion field to be well defined at the origin.

It may be pointed out that we have chosen $\sigma^2 + \vec{\pi}^2 = f_\pi^2$. More generally, we need not fix this magnitude and vary σ and $\vec{\pi}$ independently. Given such a configuration, we solve the Dirac eigenvalue equation for the quark in the background field, $\theta(r)$. Due to the linking of space with internal isospace, neither the angular momentum, $\vec{J}$, nor the isospin, $\vec{I}$, commute with the Dirac Hamiltonian but only the sum $\vec{K} = \vec{J} + \vec{I}$ does. K can then be used to label the eigenstates. The lowest, $K = 0$, valence state is a spin isopin singlet of the form

$$\psi_{K=0} = \psi_0(r)|K = 0\rangle \qquad (7)$$

where,

$$|K = 0\rangle = \frac{1}{\sqrt{2}} (|+1/2\rangle|\uparrow\rangle - |-1/2\rangle|\downarrow\rangle) \qquad (8)$$

the arrows designate spin and the halves the isospin. The eigenvalue equation is

$$(-i\partial_i \alpha_i + \beta g_y(\sigma + i\gamma_5 \vec{\tau}.\vec{\pi}))\psi_{K=0} = \epsilon_0 \psi_{K=0} \qquad (9)$$

The $|K = 0\rangle$ state is a bound state, where R has the interpretation of the width of the potential. For a particular profile, $\theta(r) = -\pi(1 - r/R)$, for $r < R$, and $\theta(r) = 0$, for $r > R$, the eigenvalue dependence on R is sketched in [1](a),[4].

The nucleon may then be obtained as a bound state of three coloured quarks. Notice that the only degeneracy of the $K = 0$ state is in colour, since this state is a spin isospin singlet and in making the nucleon we have exhausted this degeneracy, yielding a colour singlet. The energy of the nucleon state is given by

$$E^{B=1}[\theta(r)] = 3\epsilon_0[\theta(r)] + \int d^3x \frac{1}{2}[(\partial_i \vec{\pi})^2 + (\partial_i \sigma)^2] \quad (10)$$

in terms of a general profile, $\theta(r)$. This must be minimized with respect to variations of $\theta(r)$ to get the ground state energy, $E_{\min}$. Finally, since we have eigenstates of K, this quark soliton must be projected into good states of spin and isospin. The mass, M, of this soliton with quark bound states which has the quantum numbers of the nucleon depends just on f_π, g_y and λ. The dependence on λ is marginal, so the only parameter that is free is the Yukawa coupling, g_y. We fix, $M = M_{\text{nucleon}}$.

Actually, for such a nucleon, which is a colour singlet bound state of three valence quarks in a skyrme background, in a linear sigma model, a generally accepted value for $g_y = 5.4$ [1](b). This is when the π and σ fields are varied independently, without making any ansatz,

and the quark soliton so obtained is projected to give a nucleon with good spin and isospin.

Let us now compare this to the energy of 3 free quarks in the uniform phase. We find that,

$$M/(3g_y f_\pi) < 1$$

indicating that in our sigma model the nucleon is indeed a soliton with quark bound states. This analysis makes a further significant point. The threshold value of g_y at which $M/(3g_y f_\pi)$ is first less than 1, is $g_y = 4$. Therefore, for values of g_y greater than 4, the quark soliton is always of lower energy than three free quarks and is thus bound. Besides the mass of this nucleon falls with increasing g_y, as this controls the strength of the attractive potential in which the quarks bind. The conclusion, therefore, is that there is a maximum mass [18] for the fermion in such a theory.

Here we remark that we now have a determination for all parameters of our L: f_π, g_3 and g_y, leaving only one parameter, λ to be set. As we shall show this can be determined from low energy meson-meson scattering data.

Recently, Schechter et al [14] made a fit to scalar channel scattering data to see how it may be fitted with increasing $\sqrt{s}$ (centre of mass energy), using chiral perturbation theory and several resonances. They further looked at this channel using just a linear sigma model. Their results indicate that for $\sqrt{s} < 800$ MeV, a reasonable fit to the data can be made using the linear sigma model with a tree level sigma mass above but close to 800 MeV. This sets the value of λ.

This completes the determination of all the parameters of our L, which is able to describe both nucleon and quark phases of dense matter.

III. PHASES AT FINITE DENSITY

We now move to the main theme of this work, which is to look at ground state of strongly interacting matter at finite density. In our model all the phases considered are characterised simply by different patterns of spontaneous symmetry breaking. As we shall see i) the nucleonic phase is characterised by the pattern of SSB in skyrmions ii) the Lee-Wick phase has space uniform SSB, with only the sigma field expectation value and iii) the pion condensed phase is characterised by a stationary wave, with a space dependent periodic variation in the sigma and neutral pion field expectation values.

A. The nucleon or nuclear phase

In this phase, as the name suggests, the quarks are to be found as bound states in nucleons.

We found that the single isolated solitonic nucleon has lower energy than the three free valence quarks. Thus at low density we have a fermi gas of interacting nucleons. Long range potentials like pion exchange and tensor

exchange can be found by the appropriate two nucleon ansatz, in analogy to the two-skyrmion problem. These match well with the usual nuclear physics two-nucleon potentials [19]. It is then more reasonable to solve the nucleon many body problem by mapping on to the nuclear many body calculation – since it is very difficult to solve a many body quark soliton problem!

We know quite well the ground state of nuclear matter till, say, roughly two times nuclear density. From there on we can only model this phase at higher density, when the solitons start to overlap and their motion is obstructed, as a 'crystal lattice' of solitons [13]. We can then use the Wigner-Seitz approximation to convert the problem to a single cell problem. Scaling the size of the single cell scales the baryon density, since we have one nucleon per cell.

It is appropriate to note that with such simplifying approximations it is not fair to expect a realistic description but a pedagogically useful one.

The Wigner Seitz approximation

The crystal ansatz implies a picture where the solitons sit in a close packed hexagonal configuration. Each soliton is in a skyrmion chiral symmetry broken configuration for the $[\sigma, \vec{\pi}]$ fields that acts as a potential that supports a single bound state with a colour degeneracy of 3. The $|K = 0\rangle$ bound state is saturated for each soliton making the soliton into a colour singlet baryon. It is clear that when the solitons are close packed, the quark wavefunction will leak out so as to minimize the energy. From the Kronig-Penny model it is expected that bound states will form a band with the number of states in the band being $3N$, where N is the number of baryons stacked together. It is further known that the band will splay evenly about the single bound state energy, with the bottom of the band below and the top above the single soliton ($\vec{K} = 0$) bound state energy. Due to the saturation of the bound state, the band will be completely occupied and a good approximation to the median energy of the band is the bound state energy of the single soliton above of radius R. Thus the energy per baryon can be approximated in the crystal configuration by the single soliton energy but with the parameter R setting the volume occupied by a single soliton. Clearly, the baryon density is set by R.

$$n_b = \frac{6}{R^3} \qquad (11)$$

At finite density the VEV becomes an EV and we can allow for

$$< \sigma >^2 + < \pi >^2 = F^2 \qquad (12)$$

where F is now a variational parameter to be set by minimizing the free energy. It is clear that this ansatz is rather restrictive.

Note that at finite density there is just one constraint, namely

i) $< \vec{\pi} >$ must vanish at the origin of the soliton for the pion field to be well defined.

However, only for calculational simplicity we also maintain ii) and iii), below:

ii) F is space independent

iii) $< \sigma >= -F$ at $r = 0$, or
$< \sigma >= F$ at $r = R$, or
$< \vec{\pi} >= 0$ at $r = R$

We choose a simple profile function, $\theta(r) = \pi(1 - \frac{r}{R})$, where R is the half-length of the cell.

We may now calculate the Wigner-Seitz single soliton energy by calculating the quark energy eigenvalue ϵ_0, meson gradient energy (second term in eq. (13)) and symmetry (condensation) energy (third term in eq. (13)) in this background:

$$E_b = 3\epsilon_0(\theta(R)) + 2\pi F^2 R \left(1 + \frac{\pi^2}{3}\right) + \frac{1}{3}\pi\lambda^2(F^2 - f_\pi^2)^2 R^3 \qquad (13)$$

The additional complexity is that we must now minimize E_b for the single soliton energy with respect to F for each R or at each density.

Further relaxing of constraints or solving the full set of Euler-Lagrange equations for each field is somewhat laborious, so this is as far as we go. This is so because this Wigner-Seitz approximation has other uncertainties. We refer the reader to a more sophisticated treatment of this problem [20].

This section, as stated earlier, is to be seen as pedagogical. It does not yield a realistic equation of state.

Clearly the energy eigenvalue goes up as R (width of the potential) is decreased, till, at some lowest value of R, a bound state solution cannot be supported and this phase is lost.

Remarks

i) The Equation of State (EOS) with the above variational ansatz is obviously inadequate to produce the single soliton/nucleon with arbitrary variations of $< \sigma >$ and $< \pi >$. The exact solution fits the nucleon for a value of $g_y = 5.4$. This will not be the case here. We will need a larger coupling, g_y, to fit the nucleon in this approximation. Clearly, the minimum of E_b vs R, at $R_{\min}$, is the energy of a single isolated soliton.

ii) The nucleon may be obtained by projection of the soliton into good spin isospin states, $\vec{J} = \vec{I} = \frac{1}{2}$. The soliton is largely a wave packet, a linear superposition of all $\vec{J} = \vec{I} = (n + \frac{1}{2})$ states. The maximum weight comes from the lowest, $\vec{J} = \vec{I}$ states. We make the approximation that it is an equal linear superposition of the Nucleon (N) and Δ states and set the soliton energy to be midway between M_N and M_Δ.

$$E_{b,\min} = M_{\text{soliton}} = M_N + \frac{1}{2}(M_\Delta - M_N) \qquad (14)$$

iii) Even with this approximation we find that to fit $E_{b,\min}$, so as to produce a nucleon with $M_{\rm N} = 940$ MeV requires $g_y \sim 6.3$.

iv) Furthermore, there is a zero point quantum mechanical energy of localization associated with the localized nucleon states that make up the crystal that may be estimated from the Uncertainty Principle.

v) Another correction is that due to one gluon exchange interaction between the quarks.

We leave out the last two corrections and others as we shall be using a more realistic EOS anyhow (see below).

Figure 1 shows the energy per baryon, E_b versus the baryon density n_b in this phase. In this, the minimization of E_b with respect to the variational parameter F has been carried out.

As we have pointed out, to construct the exact ground state of the crystal is, to say the least, a formidable exercise. We have tried to employ an educated variational ansatz in the hope that it provides a good approximation. The EOS this yields has many correct qualitative features but nevertheless is much stiffer than most known EOS – it may not be unfair to say that it is almost arthiritic. In this instance it may be more judicious to use a well worn nuclear equation of state, like the APR98 EOS [21], for the entire nuclear phase, provided the density at which the transition to quark matter takes place is not much above twice nuclear density. A comparison between our stiff EOS and the APR98 is provided in Figure (1).

The net result is that the nucleon is a soliton, the nucleons interact with each other to produce binding at around nuclear density. We then expect to have a Fermi liquid of interacting nucleons till the nucleons begin to overlap. As the density increases the E_b increases due to topological repulsion, we expect that the nucleons are no longer free to move around and get localized into a crystal (of nucleons) like configuration analysed above. This solitonic phase then dissolves into quark matter.

IV. TWO FLAVOUR QUARK MATTER

We shall now consider in Mean Field Theory the phases of two-flavour quark matter in the $SU(2)_L \times SU(2)_R$ chiral model above.

We shall then extend the model to three flavours (u, d, s).

A. The space uniform phase

We now turn to the phase in which the pattern of symmetry breaking is such that the expectation values of the meson fields are uniform. At zero density they are just the VEVs.
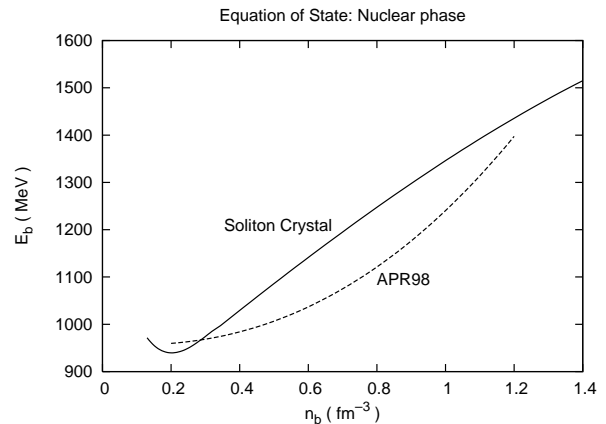
$$< \sigma > = f_\pi \qquad (15)$$

FIG. 1: Energy per baryon as a function of baryon density in the Wigner-Seitz quark soliton crystal as a model of nuclear state (solid line). The dashed line shows the corresponding relation in the APR98 [21] equation of state for beta-stable matter, obtained using A18+δv+UIX interaction model.

$$< \vec{\pi} > = 0 \qquad (16)$$

For arbitrary density we allow the expectation value to change in magnitude, as it becomes a variational parameter that is determined by energy minimization at each density.

$$< \sigma > = F \qquad (17)$$
$$< \vec{\pi} > = 0 \qquad (18)$$

Such a pattern of symmetry breaking simply provides a constituent mass to the quark $m = g_y < \sigma >= g_y F$ and the quarks are in plane wave states as opposed to the bound states in the nucleonic phase [13].

The mean field description of this phase is simple. The energy density is

$$\epsilon_\rho = \sum_{u,d} \frac{1}{(2\pi)^3}\gamma \int d^3k \sqrt{m^2 + k^2} + \frac{\lambda^2}{4}(< \sigma^2 > - f_\pi^2)^2 \qquad (19)$$

where $m = g_y < \sigma >= g_y F$ and the degeneracy $\gamma = 6$. We shall use $g_y = 5.4$ as determined from fixing the nucleon mass in this model at 938 MeV [5, 6]. The integral above runs up to the 'u' and 'd' fermi momenta.

For neutron matter (without β equilibrium) we have the relations

$$k_u^f = (\pi^2 n_u)^{\frac{1}{3}} = (\pi^2 n_b)^{\frac{1}{3}} \qquad (20)$$
$$k_d^f = (2\pi^2 n_b)^{\frac{1}{3}} \qquad (21)$$
$$E_b = \frac{\epsilon_\rho}{n_b} \qquad (22)$$

where n_b is the baryon density.

At any density the ground state follows from minimising free energy, with respect to $< \sigma >= F$. As shown in the figures of [4, 7, 13], this phase begins at $n_b = 0$, with

$E_b = 3gf_\pi$, which then falls till chiral restoration occurs at some n_X. After this, as the density is increased, E_b continues to drop, goes to a minimum and then starts rising corresponding to a massless quark fermi gas.

In the chirally restored phase the EOS is very simple and parallels the MIT bag description of [22]:

$$n_b > n_X \quad (23)$$

$$\epsilon_\rho = \left(\frac{3}{4\pi^2}\right)\pi^2 n_b^{\frac{4}{3}}\alpha + \frac{\lambda^2}{4}f_\pi^4 \quad (24)$$

The last term above is just the bag energy density, and

$$\alpha = (1 + 2^{\frac{4}{3}}) \quad (25)$$

This phase has two features, a) chiral restoration at n_X followed, with increasing density, by b) an absolute minimum in E_b, at a $n_C > n_X$.

From the comparison of this phase with the nucleon and nucleonic 'phase' arising from the same model (see [4, 13]), it is clear that the nucleonic phase is always of lower energy than the uniform phase above, upto a density of roughly 3 times nuclear density, which is above the chiral restoration density in the uniform phase.

B. The Pion Condensed phase

Here we shall consider another realization of the expectation value of $< \sigma >$ and $< \vec{\pi} >$ corresponding to pion condensation. This phenomenon was first considered in the context of nuclear matter.

Such a phenomenon also occurs with our quark based chiral σ model and was first considered at the Mean Field Level by Kutschera and Broniowski in an important paper [7]. Working in the chiral limit they found the pion condensed state has lower energy than the uniform, symmetry breaking state (phase 2) we have just considered, at densities of interest. This is expected as the ansatz for the PC phase is more general than for phase 2.

The expectation values now carry a particular space dependence

$$< \sigma > = F\cos(\vec{q}.\vec{r}) \quad (26)$$
$$< \pi_3 > = F\sin(\vec{q}.\vec{r}) \quad (27)$$
$$< \pi_1 > = 0 \quad (28)$$
$$< \pi_2 > = 0 \quad (29)$$

Note that when $|\vec{q}|$ goes to zero, we recover the uniform phase (2). The Dirac Equation in this background is solved in [13] and reduces to

$$H\chi(k) = (\vec{\alpha}.\vec{k} - \frac{1}{2}\vec{q}.\vec{\alpha}\gamma_5\tau_3 + \beta m)\chi(k) = E(k)\chi(k) \quad (30)$$

where $m = g_y F$. The interaction term has been recast in terms of the relativistic spin operator, $\vec{\alpha}\gamma_5$. It is evident that if spin is parallel to $\vec{q}$ and $\tau_3 = +1$ (up quark) then this term is negative and if $\tau_3 = -1$ (down quark) then it is positive. For spin antiparallel to $\vec{q}$ the signs for $\tau_3 = +1$ and -1 are reversed.

The spectrum for the hamiltonian is the quasi particle spectrum and can be found to be

$$E_{(-)}(k) = \sqrt{m^2 + k^2 + \frac{1}{4}q^2 - \sqrt{m^2q^2 + (\vec{q}.\vec{k})^2}} \quad (31)$$

$$E_{(+)}(k) = \sqrt{m^2 + k^2 + \frac{1}{4}q^2 + \sqrt{m^2q^2 + (\vec{q}.\vec{k})^2}} \quad (32)$$

The lower energy eigenvalue $E_{(-)}$ has spin along $\vec{q}$ for $\tau_3 = 1$, or has spin opposite to $\vec{q}$ for $\tau_3 = -1$. The higher energy eigenvalue $E_{(+)}$ has spin along $\vec{q}$ and $\tau_3 = -1$, or has spin opposite to $\vec{q}$ and $\tau_3 = +1$.

In this background the fermi sea is spin polarized into the states above. The quasi particles are, however, good states of τ_3.

First we fill up all the lower energy, $E_{(-)}(k)$, states and then we have a gap and start filling up the $E_{(+)}(k)$ states till we get to E_F^i, the fermi energy corresponding to a given density for each flavour.

$$n_i = \frac{1}{(2\pi)^3}\gamma\left(\int d^3k\Theta(E_F^i - E_{(-)}(k))\right.$$
$$\left. + \int d^3k\Theta(E_F^i - E_{(+)}(k))\right) \quad (33)$$

$$n_b = (n_u + n_d)/3 \quad (34)$$

$$\epsilon_i = \frac{1}{(2\pi)^3}\gamma\left(\int d^3k E_{(-)}(k)\Theta(E_F^i - E_{(-)}(k))\right.$$
$$\left. + \int d^3k E_{(+)}(k)\Theta(E_F^i - E_{(+)}(k))\right) \quad (35)$$

$$\epsilon_\rho = \epsilon_u + \epsilon_d + \frac{1}{2}F^2q^2 + \frac{\lambda^2}{4}(F^2 - f_\pi^2)^2 \quad (36)$$

We can now write down the equation of state as in [7]. It is found that the PC state is lower in energy than the uniform phase 2 for densities of interest. For the explicit numbers and figures we refer the reader to [7].

We briefly remark on some features of this phase.

1. The reason that the PC phase has energy lower than the uniform $< \sigma >$ condensate is perhaps best understood in the language of quarks and anti quarks. To make a condenste a quark and anti-quark must make a bound state and condense. For a uniform $< \sigma >$ condensate the q and $\bar{q}$ must have equal and opposite momentum. Therefore, as the quark density goes up the system can only couple a quark with $k > k_f$ and a $\bar{q}$ with the opposite momentum. This costs much energy so the condensate can only occur if k_f is small, at low density. On the other hand, the pion condensed state is not uniform. So at finite density, if we take a quark with $k = k_f$ the $\bar{q}$ can have momentum $k = |\vec{k}_f - \vec{q}|$, which is a much smaller energy cost

2. Since the pion condensate is a chirally broken phase, the chiral restoration shifts from very low

density in the uniform phase to very high density, $\sim 10\rho_{\text{nuc}}$. This is a signature of this phase.

3. Since this phase is lower in energy than the uniform phase for all densities of interest we go directly from the nucleonic phase to the PC phase completely bypassing the uniform phase showing that all the 'interesting' features and conjectures for the uniform phase are never realized.

4. Another feature of this π_0 condensate is that since we have a spin isospin polarization we can get a net magnetic moment in the ground state, as the magnetic moments of the u and d quarks add.

V. THE THREE FLAVOUR STATE

The extension to three flavours or $SU(3)$ chiral symmetry needs some clarification.

The generalized Dirac Equation for the $SU(3)$ case is considerably more complicated and involves a singlet ξ_0 and an $SU(3)$ octet ξ_a of scalar fields and a singlet ϕ_0 and an $SU(3)$ octet ϕ_a of pseudoscalar fields, that interact with the quarks as shown in [23].

$$H\psi(k) = (-i\vec{\alpha}.\vec{\partial} - g_y\beta(\sqrt{2/3}(\xi_0 + i\phi_0\gamma_5) + \lambda^a(\xi_a + i\phi_a\gamma_5)))\psi = E\psi \tag{37}$$

In the chiral limit, the spontaneous symmetry breaking pattern is not unique. We choose the pattern in which the $SU(3)_L \times SU(3)_R$ chiral symmetry breaks down to a vector $SU(3)$. For the uniform case, we have

$$< \xi_0 > = \sqrt{3/2}f_\pi \tag{38}$$
$$< \xi_a > = 0 \tag{39}$$
$$< \phi_0 > = 0 \tag{40}$$
$$< \phi_a > = 0 \tag{41}$$

This gives a constituent mass $m = g_y f_\pi$ for all (u, d and s) quarks. The explicit symmetry breaking strange quark mass term with current mass m_s, is then added to H. The strange quark mass, M_s, then, turns out to be the sum of the constituent and explicit mass, $M_s = g_y f_\pi + m_s$

Here we run into a puzzle of sorts. The mass term now has two components, an explicit or current mass, m_s and a constituent mass, $g_y < \sigma > = g_y f_\pi$. However, it does not disturb any relation (e.g. the GT relation), whether we choose, $< \sigma > = +$ or $-f_\pi$. This raises an ambiguity about the relative sign of the two mass terms. It would seem that the choice of opposite signs for the two terms is optimal since it gives the lowest mass or ground state energy for the one fermion sector.

When the strange quark gets a large explicit mass m_s about 150 MeV (the u and d quarks have negligible explicit masses), this question becomes very relevant. However, experimentally the strange baryons have larger masses than the non strange ones suggesting that the relative sign is plus. Georgi, in his book [24], finds that the success of the non relativistic quark model, particularly for the masses and the magnetic moments of the baryons, follows from taking constituent masses of about 350 MeV for the u and d quarks and a total mass of about 550 MeV for the strange quark. Clearly, the larger mass of strange baryons suggests the same. Thus, experimentally, the relative sign, plus, is selected. We shall therefore continue to use this.

A. The three-flavour Pion Condensed phase

For describing strange quark matter we use the 3-flavour Pion Condensed state [6]. This is a more versatile state than the one used in [22] (3 flavour Chirally Restored Quark Matter – CRQM), the latter being a subset of the former.

Next, we formulate the symmetry breaking in the presence of the pion condensate. This is given as follows,

$$< \xi_0 > = \sqrt{3/2}F(1 + 2\cos{(\vec{q}.\vec{r})})/3 \tag{42}$$
$$< \xi_8 > = -\sqrt{3}F(1 - \cos{(\vec{q}.\vec{r})})/3 \tag{43}$$
$$< \phi_0 > = 0 \tag{44}$$
$$< \phi_3 > = F(\sin{(\vec{q}.\vec{r})}) \tag{45}$$

while all other fields have zero expectation value.

This gives exactly the PC hamiltonian equation for the u, d sector and continues to give the simple mass relation for the strange quark, $M_s = g_y F + m_s$. When $q = 0$ and $m_s = 0$ we recover the chiral limit above.

We may now simply add the two-flavour PC results for the energy density and density derived above to the strange quark energy density which arises from the single particle relation,

$$E_s = \sqrt{M_s^2 + k^2}$$

The strange quark energy density is given by Baym (eqn. 8.20) [25]

$$\epsilon_s = \frac{3}{8\pi^2}M_s^4(x_s n_s(2x_s^2 + 1) - \ln(x_s + n_s)) \tag{46}$$

where $x_s = k_s^f/M_s$ and $n_s = \sqrt{1 + x_s^2}$, k_s^f being the fermi momentum of the strange quarks.

The total energy density of the quarks for the 3 flavour PC is given by

$$\epsilon_\rho = \epsilon_u + \epsilon_d + \epsilon_s + \frac{1}{2}F^2q^2 + \frac{\lambda_1^2}{4}(F^2 - (f_\pi^2))^2 \tag{47}$$

From the effective potential given in [6, 23] for the $SU(3)$ case, there is an extra factor of 3/2 that multiplies the last term. This can be absorbed, as we have done, by a redefinition, $\lambda_1 = A\lambda$, where $A = \sqrt{3/2}$.

The correction due to one gluon exchange interaction can also be incorporated in the evaluation of ϵ_u, ϵ_d and ϵ_s above. We have discussed this in some detail in [6]. In this paper we follow the prescription of Baym [25], as done in [6], to calculate the interaction contribution to quark energy densities.

B. β-equilibrium in the PC phase

We have the following general chemical potential relations for quark matter

$$E_F^u = \mu_u \tag{48}$$
$$E_F^d = \mu_d = \mu_s \tag{49}$$
$$\mu_e = \mu_d - \mu_u \tag{50}$$
$$n_e = \frac{\mu_e^3}{3\pi^2} \tag{51}$$

The charge neutrality condition below further reduces the number of independent chemical potentials to one.

$$\frac{2n_u(\mu_u, q, F) - n_d(\mu_d, q, F) - n_s(\mu_s)}{3} - n_e = 0 \tag{52}$$

The baryon density is

$$n_b = \frac{n_u(\mu_u, q, F) + n_d(\mu_d, q, F) + n_s(\mu_s)}{3} \tag{53}$$
$$n_s = (k_s^f)^3/(\pi^2) \tag{54}$$

For matter in β equilibrium we need to add the electron energy density to the quark energy density above

$$\epsilon_e = (1/4\pi^2)\mu_e^4$$

The total energy density is

$$\epsilon = \epsilon_\rho + \epsilon_e$$

The energy per baryon, $E_b = \epsilon/n_b$, then follows.

For the pion condensed state, the ground state energy and the baryon density depend on the variational parameters, the order parameter or the expectation value, $F = \sqrt{<\vec{\pi}>^2 + <\sigma>^2}$ and the condensate momentum, $|\vec{q}|$. To define the free energy at a fixed baryon density then requires some care.

We go about this by defining a baryon chemical potential to go with a baryon density. We have obtained both the baryon density and the energy density of the PC in terms of the u,d,s quark fermi energies/chemical potentials. First we construct the free energy

$$\Omega = \epsilon - n_b\mu_b = \epsilon_\rho + \epsilon_e - n_b\mu_b \tag{55}$$

TABLE I: Charge neutral, 3-flavour, beta-equilibrium pion condensed phase with $m_\sigma = 800$ MeV. The columns are: u-quark chemical potential (μ_u in MeV), baryon density (n_b in fm^{-3}), energy per baryon (E_b in MeV), electron density (n_e in fm^{-3}), ratio of densities of d-quark and u-quark (n_d/n_u), that of s-quark and u-quark (n_s/n_u), the order parameter (F in MeV) and magnitude of the vector q.

μ_u	n_b	E_b	n_e	n_d/n_u	n_s/n_u	F	q
280.0	0.2972	984.94	.2303E-02	1.916	.0318	37.0406	2.5945
300.0	0.3645	981.48	.1602E-02	1.731	.2269	31.9655	2.6149
320.0	0.4591	994.07	.1141E-02	1.599	.3640	28.6471	2.9703
340.0	0.5409	1008.88	.1173E-02	1.564	.4004	30.4335	3.0216
360.0	0.6700	1043.21	.9455E-03	1.499	.4672	28.8195	3.6217
380.0	0.7628	1062.14	.1063E-02	1.482	.4847	31.6743	3.4574
400.0	0.8967	1104.94	.4413E-03	1.345	.6245	23.6066	3.9497
420.0	1.0472	1134.64	.1283E-02	1.463	.5040	36.1496	3.9839
440.0	1.1774	1168.76	.4890E-03	1.317	.6529	26.8871	4.0720
460.0	1.3225	1216.52	.1794E-03	1.218	.7529	19.3370	4.3125
480.0	1.5149	1246.21	.3896E-03	1.267	.7033	26.9453	4.3670
500.0	1.6900	1290.81	.2212E-03	1.211	.7597	23.0806	4.4865

The baryon chemical potential is defined as

$$\mu_b = \partial\epsilon/\partial n_b \tag{56}$$

After meeting all the neutrality and equilibrium conditions above for fixed F and q, we can write all the above variables as a function of a single variable, μ_u. We then minimize Ω independently with respect to F and q. The E_b etc then follow.

The results are presented in the tables below and in Figure 2a. Fig 2b gives the the EOS for all the phases considered so far.

VI. THE PHASE DIAGRAM AND THE EQUATION OF STATE

The starting point for the phase diagram of QCD at finite density is as such: At very low density we know that there is chiral SSB, with the pion as the Goldstone boson - this breaks chiral symmetry spontaneously, leaving colour symmetry unbroken. At very high density we have a colour SC pairing instability for the quarks and of the many pairings investigated [26] the CFL (colour flavour locked) pairing is favoured – this spontaneously breaks colour and chiral symmetry. In between these limits the issue of the ground state is open.

We approach this problem, from the low density end, by the effective lagrangian L that can access both, nuclear and quark matter states, with or without chiral SSB. We have argued that this L has validity upto energy scales of about 800 MeV. The advantage of this L is that it has only 3 couplings and no other free parameters.
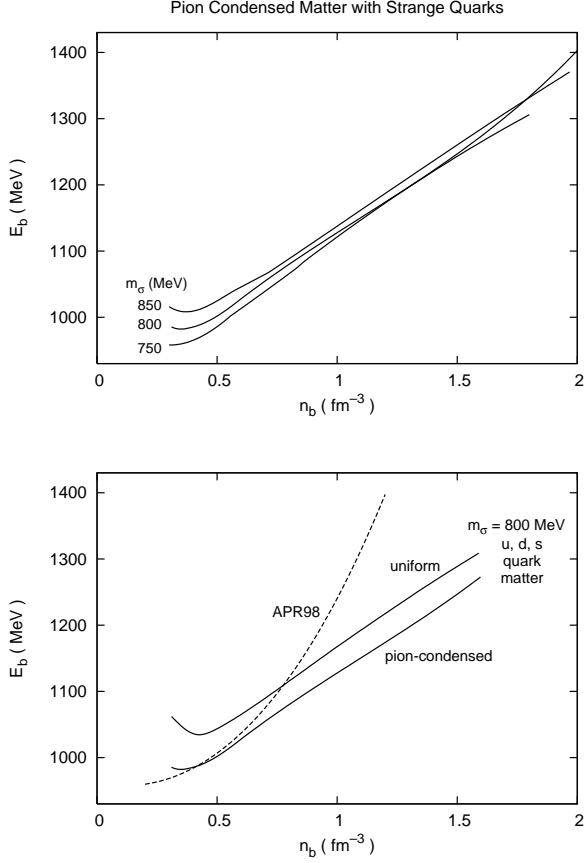
FIG. 2: (a) Upper panel: energy per baryon vs baryon number density for 3-flavour pion-condensed phase for three values of assumed tree-level mass of the scalar meson σ. Charge neutrality and beta equilibrium are imposed. One-gluon exchange interaction is included using the prescription of Baym [25]. (b) Lower panel: comparison of the equations of state of the 3-flavour space uniform phase and pion-condensed phase for $m_\sigma = 800$ MeV, with APR98 [21].

These can all be determined from experiment. However, at the mean field level, this L is good for investigating chiral condensates but not colour condensates.

If we start at the high density end, as done by the several authors who practice diquark colour superconductivity, one knows the ground state at very high density, when asymptotically free QCD makes calculation reliable. However, for intermediate density phenomenological four fermion interactions have to be introduced. This is equivalent to introducing a given value for the gap parameter, Δ [26]. Also, another phenomenological input is the bag parameter, B. Uncertainty attends these parameters. It is not clear how this ground state transits into the low density chiral SSB state.

We start with saturation nuclear matter at nuclear density and this persists till the nuclear matter gets squeezed into quark matter at moderately higher density, when a pion condensed state takes over [13]. In all these phases, as we have found, chiral symmetry is

spontaneously broken – though the patterns of symmetry breaking keep changing with baryon density. Since we have not investigated all other possible condensates, we cannot vouch for our neutral pion condensate being the best ground state.

It is, however, to be noted from the lecture notes of Baym [25], that the neutral pion condensate is preferred over the charged pion condensate for charge neutral nuclear matter, in the non relativistic limit (particularly if we put the axial coupling constant $g_A = 1$). We have found that this is the case for charge neutral quark matter as well [27]. It is worth pointing out that all these states that have lower energy than the chirally restored CRQM state, are chiral symmetry broken states.

At even higher density the most likely state is a diquark condensate - a colour flavour locked (CFL) [23] state - which can persist till arbitrarily high density. Such a ground state spontaneously breaks both chiral and colour symmetry. Diquark condensates are unlikely at moderately high density as they depend on the quark density of states and so the pion condensate is most likely at such densities.

There is an important issue that arises here and that is the comparison between the diquark condensate state and the pion condensed state. In this case the starting point is a chiral symmetric four fermion interaction which can accommodate both chiral (quark-antiquark colour singlet) condensates and diquark condensates. Such a comparison has already been done by Sadzikowski [28] in the context of a NJL chiral symmetric model, for the case of 2 flavours – $SU(2)_L \times SU(2)_R$. What is done is at the level of mean field theory. The NJL model has four fermion interactions in terms of the quark bilinears corresponding to the σ and π field quantum numbers, with a common dimensional coupling, G. If we are interested in a ground state carrying sigma and/or pion condensates we can replace these quark bilinears by the corresponding σ and π EV's in the MFT. This yields the ground state energy of the space uniform SSB and the PC states. Alternatively, this NJL can be mapped to our linear sigma model and the ground state thereof, which has been considered earlier.

To get to the diquark condensate state we have to Fierz transform this NJL chiral, L, and look for its projection into the diquark condensate channel and follow the same procedure of MFT. This projection gives the diquark condensate lagrangian used by [26, 28], with a four fermion coupling, $G' = G/4$ [28].

Working with simultaneous MF condensates, corresponding to space uniform chiral SSB (which generates a spontaneous mass for the quarks (no PC)) and diquark SC (which gives rise to colour SC), they [28] find that for the above value of the two G's, that follow from the Fierz projection, the chiral condensate is always the preferred state up till the limit of validity of this model.

However, they find that with arbitrary (higher) values of G', it is possible to have a phase transition from a space uniform chiral SSB to a diquark condensate at some

large density. In particular, they consider, $G' = G$, and find that from a pure chiral SSB state, with $m = g_y f_\pi$ = 301 MeV, there is a continuous phase transition to a mixed chiral SSB with a small admixture of diquark condensate state at a quark chemical potential of about 0.33 GeV. For the higher value of, $m = 500$ MeV we employ, this is expected to occur at a higher value of μ. This is followed by a first order phase transition at slightly higher μ, to also a dominantly diquark condensate mixed state.

This state then evolves to an almost entirely diquark condensate.

Till now we have not cosidered the PC state. The PC state always has lower energy (free energy) than the uniform chiral SSB state. It would then be reasonable to expect that the PC state and not the uniform state would be the preferred state of chiral SSB. Since chiral SSB persists till much higher density (μ) in the PC state we expect that with the inclusion of this state the transition to the colour SC state will be pushed to higher density (μ).

A following work by the same author addresses this matter by considering simultaneous MF condensates of the, PC, and the diquark condensate [29]. It is found that the first order transition does shift to higher density ($\mu = 400$ MeV). This is considered only for the particular case $G' = G/2$, and for $m = 301$ MeV. For our larger value of m this is expected to occur at larger μ. Furthermore, in this case it seems after the first order transition occurs, the chiral SSB order parameter, M, goes to zero, indicating that this state has no chiral SSB but only colour SSB. Of course, this is so as these works deal with the two-flavour case – where the colour diquark condensate is a chiral singlet.

Realistically, we must consider 3 flavours, since the quark chemical potential is much greater than the strange quark mass. In this case we are very likely to have the CFL state as the lowest energy state. The criterion for this is given in [30] and is $\Delta > m_s^2/4\mu$, which is easily satisfied. Furthermore, in this case the diquark condensate is a colour-flavour condensate which has both chiral SSB and colour SSB, albeit in a manner different to the PC.

The deciding question is then what effective L is to be used. There is no derivation of the exact effective L from QCD at arbitrary scales. There are many four fermion type (or higher order) interactions derived or rather motivated from various points of view, e.g. instantons, gluon exchange etc. However, there is no unique L that is derived as such. It is therefore necessary to have an argument to decide this issue. See also [34] for a related discussion.

Since the NJL chiral L identifies with our linear sigma model, and this latter model is what we have used as a valid model till centre of mass energies/scales of less than 800 MeV, the right procedure would be to take this model to describe physics upto this scale. In this case, as we have argued, the PC is the preferred ground state, till the scale of validity of our effective L (this is for $G' = G/4$).

Even if we relax this to larger G' we have a three-flavour PC, till μ well above 400 MeV, followed by a CFL state, with increasing density. As the tables suggest such values of, μ, correspond to baryon density 5–6 times nuclear density. This makes the PC as likely state in neutron star cores.

With increasing baryon density we then find the following hierarchy. At nuclear density and above we have nuclear matter with chiral SSB, followed by the pion condensed quark matter, again with chiral SSB, albeit with a different realization and finally a transition to the diquark CFL state which also has chiral SSB (and colour SSB), with yet another realization. A point to note is that CRQM or free fermi seas are unstable. To one's surprise at zero temperature, at any finite density chiral symmetry is never restored! A similar result has also been obtained in 1+1 dimension [35].

VII. DISCUSSION AND IMPLICATIONS FOR NEUTRON/STRANGE STARS

A. Stars

We have used the quark based linear sigma model for this analysis, where the tree level sigma mass was set to be around 800 MeV as indicated by the analysis of Schechter et al [14], by matching the results for meson-meson scattering from this model, with experiment.

Such a tree level sigma mass also ruled against conventional SQM being the ground state of matter [6]. Strange stars require the absolute stability of SQM. Since this has been shown to be highly implausible so is then the existence of strange stars.

Independently, we find (Bhattacharya and Soni, in preparation) that realistic Neutron stars with PC cores occur only when the tree level sigma mass in this model is in a small window, 750 - 850 MeV.

This can be seen from E_b vs $1/n_b$ diagram for the phases we have considered (Fig 3). The negative slope in this figure gives the pressure and the intercept on the vertical axis the baryon chemical potential. The Maxwell contruction of a common tangent to these curves gives the pressure and baryon chemical potential at which the transition occurs with a density discontinuity.

We find that at $m_\sigma = 800$ MeV we get a transition with a relatively small pressure and a small density discontinuity. This permits us to have a well developed PC core in a neutron star. The small density jump, besides, reassures us that a mixed [31] phase would give results that would be very similar. Of course, with the APR98 [21] EOS that we use, we do not have analytical expressions for all variables, which precludes a mixed phase calculation.

At $m_\sigma = 850$ MeV, the transition moves to higher density and higher pressure with a larger density jump, so much so that there there is no PC core for the star – as the maximum mass instability for the star occurs before
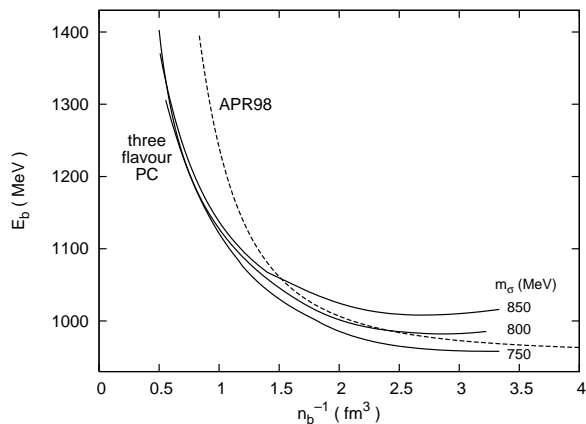
FIG. 3: The Maxwell construction: Energy per baryon plotted against the reciprocal of the baryon number density for APR98 equation of state (dashed line) and the 3-flavour pioncondensed (PC) phase, for three different values of m_σ (solid lines). A common tangent between the PC phase and the APR98 phase in this diagram gives the phase transition between them. The slope of a tangent gives the negative of the pressure at that point, and its intercept gives the chemical potential. As this figure indicates, the transition pressure moves up with increasing m_σ, and at m_σ below ~ 750 MeV a common tangent between these two phases cannot be obtained.

a core can form.

On the other side, at $m_\sigma = 750$ MeV, the curves for the two phases do not permit a common tangent construction any more – so again a star with a PC core is ruled out. Is this a coincidence that a single parameter in our effective L, the mass of the sigma or λ, plays a crucial role? Is it fortuitous that the tree level sigma mass set by scattering experiments sits in a small window that simultaneously rules out SQM as the absolute ground state of matter and also can provide us with neutron stars that can have pulsar range magnetic PC cores?

The problem in sustaining a PC core with a nuclear exterior is that we have a stiff exterior with a soft interior – a rather unstable situation. It is thus not so surprising that very particular conditions must obtain for this to occur.

B. The Effective Lagrangian and Mean Field Theory

The question of the validity of our effective L is of the essence. Furthermore, since we do not go any further than MFT, the question of the validity of MFT is another equally important question. Linked to this is the question of QCD corrections.

1. Meson sector

Let us begin with the part of the Lagrangian that describes the meson sector. As we have said earlier the justification for this, its range of validity and the value of

the coupling λ or the sigma mass have been very clearly spelt out in [14]. It will be useful to review this. As stated in this work it is the tree level Lagrangian that is used to describe π-π scattering in the scalar channel. This tree-level amplitude is then improved by unitarising the K matrix. A very good fit to scattering data in this channel follows if we choose the sigma mass in the tree level L to be 800–850 MeV. The validity of this vis a vis the data is upto $\sqrt{s} < 800$ MeV.

The actual mass and width of the sigma can then be gleaned by looking at the pole in this K unitarized amplitude and is found to be 460–600 MeV.

We have used this to fix the tree level coupling/sigma mass for our L. Since we work only in MFT (tree level), this also perhaps tells us that we may be off the mark by about 30%.

Another cautionary point is that the above results are based only on scalar channel scattering and may not apply generally. It is also possible to fit the data with chiral perturbation theory (infinite m_σ) and the ρ resonance. It is possible that by excluding the ρ we may be missing some short-distance repulsion, which might result in making our PC EOS too soft.

2. Quark sector

For obvious reasons one cannot find an analagous scattering experiment for the quark meson sector. However, there is other indirect evidence in this sector.

i) The quark soliton nucleon that we have used to fix the coupling, g_y, satisfies the Goldberger Triemann relation exactly [32] in accord with chiral symmetry. As stated in the introduction several independent properties of this nucleon, for example, magnetic moments compare well with experiment.

ii) In the large N_c expansion, the soliton (and other physics) receives $1/N_c$ corrections from loops.

iii) Goksch [12] finds that the finite temperature screening masses of mesons computed in lattice QCD compare very favorably with those calculated using finite temperature MFT for precisely our L (without including gluons). However, the couplings used are slightly different to ours, $g_y = 3.3$ and sigma mass of 600 MeV.

Georgi and Manohar [3] use a non linear sigma model version of the above L to do effective cutoff field theory and argue for larger energy scale of chiral symmetry breaking vis a vis confinement. Using this scale for the cutoff, they are able to naively but consistently include arbitrary meson/quark loops. Further, they heuristically argue that the QCD coupling is weak for the chiral SSB vacuum and thus gluon loops may be ignored.

Actually, as we have stated in the introduction, QCD can have multiple scales [5] - a confinement scale, a chiral symmetry restoration scale and a compositeness scale for the pion. We would like to point out that the arguments of [3] would go through if the compositeness scale

was substituted for the chiral restoration scale, since this scale is used for a momentum cutoff. This will allow the chiral restoration scale to fall anywhere in between the confinement and compositeness scale – a likely possibility suggested by the lattice.

Parenthetically, we remark that the non-perturbative input of a finite mass sigma particle/resonance may do the same job. This is also supported by observations in Ref. [14] and references therein. This also supports our effective L, which is built on the premise that the chiral SSB energy scale is larger than than the confinement scale.

3. QCD sector

We have no evident justification for neglecting this strong sector or at most doing simply one gluon exchange corrections.

Ref. [3] makes the heuristic argument that the gluon coupling becomes strong to precipitate chiral SSB, after which it is expected to be perturbative in the new broken vacuum. They also argue, that from the colour and electromagnetic hyperfine splittings in the baryon sector of the non relativistic quark model one may infer an $\alpha_s = 0.28$.

At finite density, there is screening in quark matter. The correct expansion parameter is then the screened charge and not the usual running coupling constant written as a function of the quark chemical potential which is commonly used. Such a screened charge has been formally constructed by one of us [33] and is clearly much smaller than the usual running coupling. This makes a perturbative expansion in this coupling more plausible for dense quark matter.

Further, the ground state of quark matter, the pion condensate, has spin-charge ordering. Long range forces like one gluon exchange then inhibit any change in the ground state and stabilize it against fluctuations as in many condensed matter situations where the coulomb forces are operative. In this case MFT may be valid.

All the calculations referred to above for quark matter states have been carried out in MFT. We have tried to give some justification for its use. However, we cannot provide any rigorous proof for the validity of MFT. It is also the simplest thing to do and works well in many cases even if the coupling is strong.

Since we began this work with an eye to neutron stars it may be appropriate to present our finding. We expect that the density profile of the star will start with the nucleonic EOS on the outside and go to a pion condensate in the interior and could well go to a colour flavor locked state at the centre if the density there is large enough; though, this is unlikely if we have a pion condensate as its softness may result in a smaller maximum mass.

Acknowledgements

[1] (a) S. Kahana, G. Ripka and V. Soni, Nucl. Phys. A415, 351 (1984), (b) M. C. Birse and M. K. Banerjee, Phys. Lett. 134B, 284, (1984).

[2] A. Manohar and H. Georgi, Nucl. Phys. 234B, 203 (1984).

[3] V. Soni, Mod. Phys. Lett. A, Vol. 11, 331 (1996).

[4] V. Soni, 'The nucleon and strongly interacting matter', Invited talk at DAE Symposium in Nuclear Physics, Bombay, Dec 1992 and references therein.

[5] V. Soni, M. S. Modgil and D. Sahdev, Phys. Lett. 552B, 165 (2003).

[6] V. Soni and D. Bhattacharya, Phys. Rev. D69, 074001 (2004).

[7] M. Kutschera, W. Broniowski and A. Kotlorz, Nucl. Phys. A516 566 (1990).

[8] M. C. Birse, Soliton Models in Nuclear Physics, Progress in Particle and Nuclear Physics, Vol. 25, 1 (1991), and references therein.

[9] see for example, R. Johnson, N. W. Park, J. Schechter, V. Soni and H. Weigel, Phys. Rev. D42, 2998 (1990), J. Stern and G. Clement, Mod. Phys. Lett, A3, 1657 (1988).

[10] see for example, J. Stern and G. Clement, Phys. Lett. 264B, 426 (1991), E. J. Eichten, I. Hinchcliffe and C. Quigg, Fermilab-Pub 91/272-T.

[11] D. Diakonov, V. Petrov, P. Pobylitsa, M. Polyakov and C. Weiss, Nucl. Phys. B480, 341 (1996), Phys. Rev. D56, 4069 (1997).

[12] A. Gocksch, Phys. Rev. Lett. 67, 1701 (1991).

[13] V. Soni, Phys. Lett. 152B, 231 (1985).

[14] A. Abdel-Rehim, D. Black, A.H. Fairbroz, S. Nazri and J. Schechter, Phys. Rev D68, 013008 (2003).

[15] V. Soni (unpublished)

[16] V. Soni, Phys. Lett. 183B, 91 (1987), Phys. Lett. 195B, 569 (1987).

[17] see also related work, R. Perry, Phys. Lett. 199B, 489 (1987), G. Ripka and S. Kahana, Phys. Rev. D36 1233 (1987), W. Broniowski and M. Kutschera, Phys. Lett. 234B, 449 (1990).

[18] V. Soni, Nucl. Phys. B315, 121 (1989) and The sigma model nucleon - Yukawa or topological, Proc. of the 2nd Hellenic School on Elementary Particle Physics, Ed E. N. Argyres and G. Zoupanous, 430, World Scientific, (1985).

[19] J. Segar, M. Sripriya and M. S. Sriram, Int. J. Mod. Phy. E3, 769 (1994), R. Vinh Mau, M. Lacombe, B. Loiseau, W. N. Cottingham and P. Lisboa, Phys. Lett. 150B, 259 (1985), R. Vinh Mau, M. Lacombe, B. Loiseau and W. N. Cottingham, Phys. Lett. 161B, 31 (1985).

[20] I. Klebanov, Nucl Phys B 262, 133 (1985), U. Weber and J. A. Mcgovern, Phys. Rev. C 3376 (1998).

[21] A. Akmal, V. R. Pandharipande and D. G. Ravenhall, Phys. Rev. C 58 1804 (1998).

[22] E. Farhi and R. L. Jaffe, Phys. Rev. D 30, 2379 (1984).

[23] J. A. McGovern and M. C. Birse, Nucl. Phys. A506, 367 (1990).

[24] H. Georgi, in Weak interactions and Modern Particle Theory, Benjamin/Cummings (1984).

[25] G. Baym, in NORDITA lectures, 'Neutron Stars and the Properties of Matter at high density', (1977). See also, G. Baym and S. Chin, Phys. Lett. 62B, 241 (1976).

[26] M. Alford, K. Rajgopal and F. Wilczek, Phys. Lett. 422B, 247 (1998), Nucl. Phys. B537, 443 (1999).

[27] V. Soni and D. Bhattacharya, unpublished.

[28] M. Sadzikowski, Mod. Phys. Lett. A 16, 1129 (2001).

[29] M. Sadzikowski, Phys. Lett. B553, 45 (2003).

[30] M. Alford, K. Rajgopal, S. Reddy and F. Wilczek, Phys. Rev D 64, 074017 (2001).

[31] V. Soni, The collective quantization of a chiral soliton coupled to quarks, University of Regesburg preprint, TPR85-29 (1985).

[32] N. K. Glendenning, Compact Stars, Nuclear Physics, Particle Physics and General Relativity, Springer Verlag, New York, (1997).

[33] V. Soni, Nucl. Phys. B 216, 244 (1983), B 216, 267 (1983), for a review see, Proceedings of the Int'l Conf. on the Physics and Astrophysics of the Quark Gluon Plasma, World Scientific, 73 (1988).

[34] S. B. Ruster, V. Werth, M. Buballa, I. A. Shovkovy and D. H. Rischke, hep-ph/0503184 (2005).

[35] V. Schön and M. Thies, Phys. Rev. D62, 096002 (2000). O. Schnetz, M. Thies, K. Urlichs, Annals Phys. 314, 425 (2004).